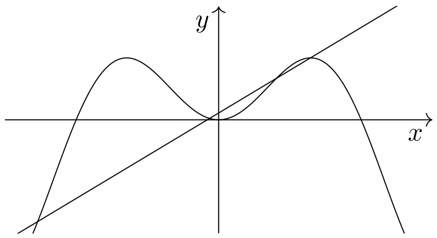
2401. True or false? For a sample coded by $y_i = ax_i + b$,

(a) $\bar{y} = a\bar{x} + b$,

(b) $s_y = as_x + b$,

(c) $\text{IQR}_y = a\,\text{IQR}_x$.

2402. The graphs shown below are $y = 5x \sin x$ (defined in radians) and $y = 4x + 1$.



By making a suitable approximation, determine to 1dp the x coordinate of the point of intersection near the origin.

2403. Two particles, masses m and $2m$, exert a constant force on each other for $t \in [0, t_0]$. They accelerate, both starting from rest. Afterwards, both particles travel at constant speeds v and kv.

(a) Show that $k = \frac{1}{2}$.

(b) Write a_1 and a_2 in terms of v and t_0.

(c) Show that, at time $t \geq t_0$, distance between the particles is given by $d = \frac{3}{4}v(2t - t_0)$.

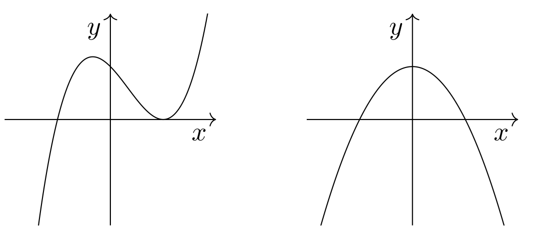
2404. A positive cubic function g has, for some $\alpha \in \mathbb{R}$, $\text{g}(\alpha) = 0$, $\text{g}'(\alpha) = 0$, $\text{g}''(\alpha) = 0$. Sketch the graph $y = \text{g}(x)$ for x values close to α.

2405. If $\dfrac{d}{dx}\left(\dfrac{1}{x+y}\right) = 0$, show that $\dfrac{dy}{dx} = -1$.

2406. For constants a, b, c, d, either prove or disprove the following formula:

$$\int_a^b y\,dx + \int_c^d y\,dx = \int_a^d y\,dx + \int_c^b y\,dx.$$

2407. A cubic function $x \mapsto \text{f}(x)$ and quadratic function $x \mapsto \text{g}(x)$ have graphs as shown. Each has either a single or double root at $x = \pm 1$.



Sketch $y = \text{f}(x)\,\text{g}(x)$.

2408. Show that the following statement cannot be an identity for any $A, B \in \mathbb{R}$:

$$\frac{x^2}{x^2 - 1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

2409. An exponential model of cooling is

$$T = A + B \times 2^{\lambda t},$$

where T is temperature, t is time, and A, B, and λ are constants. The (t, T) data points $(0, 240)$, $(1, 180)$ and $(2, 165)$ are given.

(a) Write down three equations in A, B, λ.

(b) Using substitution, show that

$$-60 = B(2^{\lambda} - 1),$$
$$-75 = B(2^{2\lambda} - 1).$$

(c) Find the values of A, B, λ.

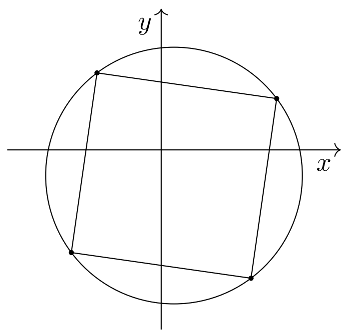
(d) Find the predicted long-term behaviour.

2410. If $y = \operatorname{cosec} u$, show that $y\dfrac{du}{dy} + \tan u = 0$.

2411. (a) Sketch the graphs $y = |x^2 - 1|$ and $y = x^2$ on the same set of axes.

(b) Hence, solve the equation $|x^2 - 1| = x^2$.

2412. Four points $(-5, 6)$, $(9, 4)$, $(-7, -8)$, $(7, -10)$ are the vertices of a cyclic quadrilateral.



Show that the diagonals of the quadrilateral bisect each other.

2413. Solve the inequality $(2x + 1)^3 - (2x - 1)^3 > 98$.

2414. Consider a pair of functions f and g, each defined for all real numbers.

- The set of x values satisfying both $\text{f}(x) = 0$ and $\text{g}(x) = 0$ is P.
- The set of x values satisfying both $\text{f}(x) \neq 0$ and $\text{g}(x) \neq 0$ is Q.

State, with a reason, whether the following hold:

(a) $P \cup Q = \mathbb{R}$,

(b) $P \cap Q = \varnothing$.

2415. Disprove the following statement: "The mean of two prime numbers cannot be a prime number."

2416. Using a suitable trigonometric identity, determine the exact value of $\sin 15°$.

2417. The parabola P has equation $y = -x^2 + 6x + 4$. P has a maximum at point (a, b). Find the equation of the parabola which is a translation of P and has a maximum at point $(-a, b)$.

2418. By expressing the integrand as a proper fraction, or otherwise, show that

$$\int \frac{4x}{2x-1}\, dx = 2x + \ln|2x-1| + c.$$

2419. Simplify the following sets:

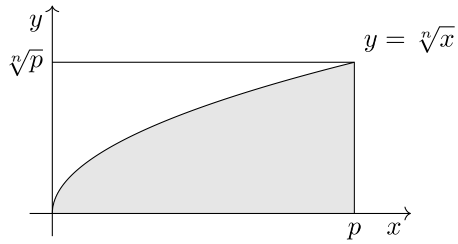
(a) $[0, 1) \cap (0, 1]$,

(b) $[0, 2] \setminus [1, 2]$,

(c) $(0, 1) \cup \{0, 1\}$.

2420. Prove that, if the third difference of a sequence is a non-zero constant, then the sequence is cubic.

2421. An (x, y) rectangle is drawn, with vertices at $(0, 0)$, $(p, 0)$, $(p, \sqrt[n]{p})$, $(0, \sqrt[n]{p})$, for some $p > 0$ and $n \in \mathbb{N}$.



Show that the curve $y = \sqrt[n]{x}$ divides the area of this rectangle in the ratio $1 : n$.

2422. Show that the graph $y = 10^x$ may be transformed into $y = e^x$ by a stretch in the x direction.

2423. At the points where $f(x) = -1$, determine if the quartic function $f(x) = x^2(x^2 - 2)$ is increasing, decreasing or neither.

2424. Prove that, if natural numbers m and n differ by two, then $m^2 + n^2$ cannot be prime.

2425. In this question, do not use a polynomial solver.

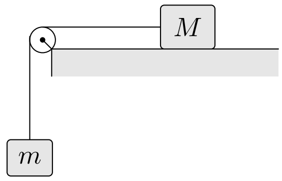
Use a numerical method to find the two real roots of the following equation, to 3sf:

$$\frac{1}{(1+\sqrt{x})^4} + \frac{1}{(1-\sqrt{x})^4} = 1.$$

2426. State, with a reason, a general formula for

$$\int f'(x)\big(f(x)\big)^n\, dx.$$

2427. Two blocks, masses M and m kg, are connected by a light string. This is passed over a smooth, light, fixed pulley, as shown in the diagram. The block with mass M kg sits on a smooth tabletop.



(a) State a further assumption required to model the blocks as having the same acceleration.

(b) Show that $a = \dfrac{mg}{M+m}$.

2428. Simplify $\log_{ab} a^n + \log_{ab} b^n$.

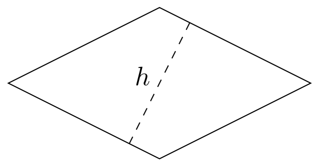
2429. A cone is undergoing a transformation. When its radius and perpendicular height have the same value x cm, its radius is increasing at $\frac{1}{4}x$ cm/s and its height is increasing at $\frac{1}{3}x$ cm/s. Show that the rate of change of the volume of the cone at this moment is $\frac{5\pi}{18}x^3$.

2430. Show that the following function is well defined over the domain $[1, \infty)$:

$$f : x \longmapsto \frac{1}{\sqrt{5x^2 - 3x - 1}}.$$

2431. Simplify $\displaystyle\int \frac{3x-1}{3x-2}\, dx$.

2432. The diagram shows a rhombus, with h defined as the height perpendicular to a pair of parallel sides:



Prove that the area of the rhombus is given by $A = h^2 \operatorname{cosec}\theta$, where θ is any interior angle.

2433. One of the following statements is true; the other is not. Identify and disprove the false statement.

① $x^3 + 4x^2 + 3x = 0 \implies x = 0$,
② $x^3 + 3x^3 + 3x = 0 \implies x = 0$.

2434. Write the following in terms of $\log_3 x$:

(a) $\log_{27} x^2$,

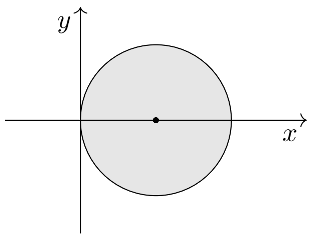
(b) $\log_{\frac{1}{3}} \frac{1}{\sqrt[3]{x}}$.

2435. A boy of mass 25 kg is standing on a sledge of mass 10 kg. The sledge is on smooth, flat snow. The boy jumps from the sledge, exerting an extra force, in addition to the usual reaction, of 400 Newtons, at $60°$ below the horizontal. This force is modelled as constant for the 0.25 seconds of take-off.

(a) Show that the boy leaves the sledge travelling at 4 ms^{-1}, having moved 50 centimetres.

(b) Find the speed of the sledge after the jump.

2436. A point (x, y) is randomly chosen in the interior of the circle $(x-1)^2 + y^2 = 1$, as shaded below:



Find the probability that the coordinates of this point satisfy $y \geq \sqrt{3}\,(x-1)$.

2437. In each case, a pair of quantities is listed, which refer to a geometric sequence $u_n \in \mathbb{R}$. State, with a reason, whether knowing the listed quantities would allow you to calculate, with certainty, the hundredth term of the sequence.

(a) First term; second term.

(b) First term; third term.

(c) First term; fourth term.

(d) First term; fifth term.

2438. Prove that, for any triangle, there is exactly one circle which passes through all three vertices.

2439. This equation is known as the *sophomore's dream*:

$$\sum_{n=1}^{\infty} n^{-n} = \int_0^1 x^{-x}\, dx.$$

Taking the value of 0^0 to be 1, approximate the two sides of the identity with

(a) LHS: the sum to $n = 10$,

(b) RHS: a trapezium rule with four strips.

2440. Find the sum of the integers from 1 to 100 which are not multiples of 5.

2441. In this question, do not use a calculator.

A tangent is drawn to $y = 8\cos^3 x - 1$ at $x = \frac{\pi}{3}$. Determine the equation of the tangent line, giving your answer in the form $y = \sqrt{a}(\pi - bx)$, where $a, b \in \mathbb{N}$.

2442. The vertices of triangle ABC have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ relative to an origin O. The midpoints of the edges opposite A, B, C are labelled P, Q, R.

(a) Write down the position vectors of P, Q, R.

(b) Determine the position vector of

i. X, which divides AP in the ratio $\lambda : 1-\lambda$, for $\lambda \in [0, 1]$

ii. Y, which divides BQ in the ratio $\mu : 1-\mu$, for $\mu \in [0, 1]$

(c) By solving simultaneous equations, find λ and μ such that X and Y coincide.

(d) Prove that AP, BQ and CR are concurrent.

2443. Write down the period of each of the following functions, defined in radians:

(a) $\text{f}(x) = \cos 3x$,

(b) $\text{g}(x) = \sin 2x + \cos 5x$,

(c) $\text{h}(x) = \sin 4x + \cos 8x$.

2444. Show that $\dfrac{d}{dx}\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \dfrac{1}{(1-\sqrt{x})^2\sqrt{x}}$.

2445. The *sigmoid* function is defined over $\mathbb{R}$ as

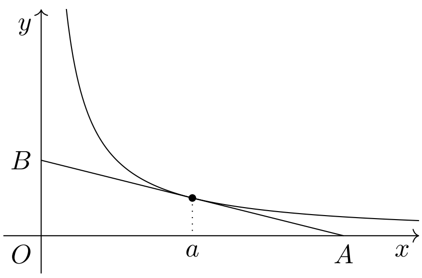
$$\text{S}(x) = \frac{e^x}{e^x + 1}.$$

(a) Prove that $\text{S}(x) \equiv 1 - \text{S}(-x)$.

(b) Find $\displaystyle\int \text{S}(x)\, dx$.

2446. Solve $\ln(\sin x) + \ln(\cos x) + \ln 2 = 0$ for $x \in [-\pi, \pi]$.

2447. The curve $y = x^{-1}$ has a tangent drawn to it at $x = a$, for $a > 0$. This tangent crosses the x axis at A and the y axis at B.



Show that the area of $\triangle AOB$ is independent of the value of a.

2448. A student makes the following claim in a proof: "No natural number of the form $n^2 - 1$, for $n \in \mathbb{N}$, can be prime, because $n^2 - 1$ can be factorised as a difference of two squares, according to

$$n^2 - 1 \equiv (n+1)(n-1)."$$

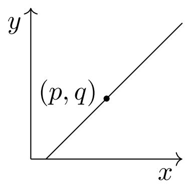
Criticise this claim.

2449. Determine the equation of the normal to the curve $y = \tan x + \cot x$ at $x = \frac{\pi}{4}$.

2450. In this question, assume the Earth to be a sphere of mass 6×10^{24} kg. A person of mass 60 kg jumps vertically, achieving a greatest displacement of 1 metre. The jump is modelled with a constant NIII pair, acting on rigid objects, lasting for $\Delta t = 0.4$ seconds. Find

(a) the take-off speed of the person,
(b) the magnitude of the NIII pair,
(c) the speed of the Earth at take-off,
(d) the gravitational acceleration of the Earth while the person is in the air,
(e) the greatest displacement of the Earth during the motion.

2451. The diagram shows a straight line of gradient 1, passing through the point (p, q), where $p > q > 0$. The equation of the line is given.



$$\frac{y - q}{x - p} = 1$$

Sketch the following graphs, for $p > q > 0$:

(a) $\dfrac{|y - q|}{x - p} = 1$,
(b) $\dfrac{y - q}{|x - p|} = 1$.

2452. The function f is $f(x) = a + be^{cx+d}$, for constants a, b, c, d. It is an increasing function for all $x \in \mathbb{R}$. State, with a reason, whether the following hold:

(a) The equation $f(x) = 0$ has at most one root.
(b) The equation $f(x) = x$ has at most one root.
(c) The equation $f(x) = -x$ has at most one root.

2453. From a large population, a sample of five is taken. Find the probability that at least one of the five is above the upper quartile.

2454. A line has parametric vector equation

$$\mathbf{r} = 5\mathbf{i} + t(\mathbf{i} + 3\mathbf{j}).$$

Determine, in the same form, the new equation of the line, if it is

(a) reflected in $y = x$,
(b) rotated 90° clockwise around the origin.

2455. Disprove the following: "If a graph $y = g(x)$ has a local maximum at (a, b) and is stationary nowhere else, then it intersects the line $y = b$ exactly once."

2456. Factorise $1170a^2 - 389ab - 165b^2$.

2457. Vectors $\mathbf{p}$ and $\mathbf{q}$ are defined as

$$\mathbf{p} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$
$$\mathbf{q} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

(a) Give the magnitudes of $\mathbf{p}$ and $\mathbf{q}$.
(b) Show that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.
(c) Sketch $\mathbf{p}$ and $\mathbf{q}$ for $\theta = -\frac{1}{6}\pi$.
(d) Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{p}, \mathbf{q}$ and θ.

2458. Prove that, for $b \neq 0$, $\displaystyle\lim_{x\to\infty} \frac{1 + ax^2}{1 + bx^2} = \frac{a}{b}$.

2459. State, with a reason, whether the following are valid identities:

(a) $|\cos x| \equiv \cos x$,
(b) $\cos |x| \equiv \cos x$.

2460. A *Sophie Germain* prime p is one for which $2p+1$ is also prime. The first four Sophie Germain primes are $2, 3, 5, 11$.

Prove that a Sophie Germain prime $p > 3$ must leave a remainder of 2 when divided by 3.

2461. Prove that $2x - 3y > 5 \implies x^2 + y^2 > 1$.

2462. A function f is defined over the reals, and has range $[-2, 2]$. Give the ranges of the following, each over the largest possible real domain:

(a) $x \longmapsto \dfrac{1}{f(x) + 1}$,
(b) $x \longmapsto \dfrac{1}{f(x) + 2}$,
(c) $x \longmapsto \dfrac{1}{f(x) + 3}$.

2463. Solve $x^3 - x^{\frac{3}{2}} = 56$.

2464. In a crossword, the following letters have been filled in of a six-letter clue:

G		D		L	

The solver knows that the letter between G and D must be one of the five vowels. Find the number of possible combinations of letters if

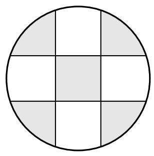
(a) there are no other restrictions,
(b) the six letters must all be different.

2465. In this question, do not use a calculator.

Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 - \cos 2x \, dx$.

2466. Either prove or disprove the following statement: "There are sets of two linear equations in x and y which are satisfied by infinitely many (x, y) pairs."

2467. An icon is designed as below, where the length of each of the twelve straight line segments is l.



Show that the area of the circle is $\dfrac{5\pi l^2}{2}$.

2468. Find any stationary points of $y = \dfrac{100x}{100 + x^2}$.

2469. Write the following as $a\sqrt[3]{2} + b$, where $a, b \in \mathbb{Z}$:

$$\left(1 + \sqrt[3]{2}\right)^3 - \left(1 - \sqrt[3]{2}\right)^3.$$

2470. A student writes: "When a truck goes around a roundabout, the truck exerts a frictional force on the surface of the road. This force acts outwards, away from the centre of the roundabout." Explain whether this is correct.

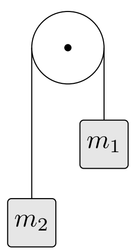
2471. Show that, if the function $g(x) = x^3 + kx^2 + kx$ is increasing everywhere, then $k \in (0, 3)$.

2472. The (x, y) plane is transformed in such a way that the following graphs map to one another:

$$y = ae^{px+c} \longmapsto y = be^{qx+d}.$$

Determine the area scale factor.

2473. Two blocks of masses m_1 and m_2, where $m_1 > m_2$, are attached to opposite ends of a length of light, inextensible string, which is passed over a fixed pulley.



Show that, if resistances are ignored, the common acceleration a ms^{-2} of the two blocks is given by

$$a = \frac{m_1 - m_2}{m_1 + m_2} g.$$

2474. Show that $y = \sin(\arccos x)$ is a semicircle.

2475. Functions f and g are each defined over the reals. The solution sets for the equations $f(x) = 0$ and $g(x) = 0$ are denoted A and B. You are given that $A \cap B = \varnothing$.

For each of the following, write down, in terms of A and B, the solution set of the equation, or state that there is not enough information to do so:

(a) $f(x)\,g(x) = 0$,

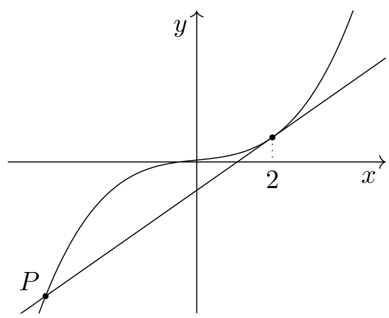
(b) $fg(x) = 0$,

(c) $\dfrac{f(x)}{g(x)}$.

2476. Show that the curves $y = x^2 + x$ and $x = y^2 + y$ are tangent, and sketch them on one set of axes.

2477. In this question, do not use a calculator.

A tangent is drawn to the cubic $y = x^3 + 2x + 1$ at the point with x coordinate 2.



(a) Find the equation of this tangent.

(b) Show that, at the point P where this tangent crosses the curve again, $x^3 - 12x + 16 = 0$.

(c) Hence, find the coordinates of P.

2478. The variable X has distribution $N(0, 1)$.

(a) Explain why, for any positive constant k,

$$\mathbb{P}(X > k) = \mathbb{P}(X < -k).$$

(b) Write down the values of

i. $\mathbb{P}(0 < X < k \mid X^2 > k^2)$,

ii. $\mathbb{P}(X < 0 \mid X^2 > k^2)$.

2479. Two quadratic functions f and g have $f'(p) = g'(p)$ and $f'(q) = g'(q)$ for some constants $p \neq q$. Prove that $f(x) - g(x)$ is constant.

2480. Find the range, over the reals, of

(a) $\sin|x|$,

(b) $|\sin x|$.

2481. You are given that the following curve is stationary at $x = 0$:

$$y = \frac{x^2 - p}{x^2 - q}.$$

Determine what, if anything, can be said about the constants p and q.

2482. Show that the solution set of $\sin x - 1 = 4\cos^2 x$ is $\{x : x = 90° + 360°n, n \in \mathbb{Z}\}$.

2483. A cricket bowling machine fires balls horizontally, from a height of 2.5 m, at 20 ms^{-1}, with the speed varying by up to 2%. Ignoring air resistance, find the possible distances d from the bowling machine to the points at which the balls land.

2484. Four values x_1, x_2, x_3, x_4 are chosen at random and independently from the interval $[0, 1]$. Write down the probability that $x_1 < x_2 < x_3 < x_4$.

2485. By considering the vertex of a parabola, find the range of the function $\text{g}(x) = x^2 - 4x$ on the domain $(0, k)$, distinguishing between the cases

(a) $k \in (0, 2)$,

(b) $k \in [2, 4)$,

(c) $k \in [4, \infty)$.

2486. A GP has the formula $u_{n+1} = \frac{3}{2}u_n$, with $u_1 = 5$.

(a) Write down the ordinal formula.

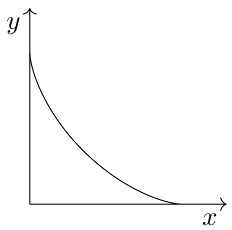
(b) Determine the smallest value of n for which the difference between the n^{th} and $(n+1)^{\text{th}}$ terms is greater than 1000.

2487. Consider the function $\text{f}(x) = x^x$.

(a) Write this as an exponential with base e.

(b) Hence, show that $\text{f}'(x) = x^x(\ln x + 1)$.

2488. Below is a sketch of $y = \text{g}(x)$, where g has domain and range $[0, 1]$.



Sketch the following graphs, including negative values of x and y where appropriate:

(a) $y = \text{g}(|x|)$,

(b) $|y| = \text{g}(x)$,

(c) $|y| = \text{g}(|x|)$.

2489. Solve $6\cos^2 x + \sin x - 5 = 0$, for $x \in [0, 360)°$.

2490. Variables $x_1, x_2, ..., x_5$ are related as follows:

$$\frac{dx_i}{dx_{i+1}} = i, \quad \text{for } i = 1, 2, 3, 4.$$

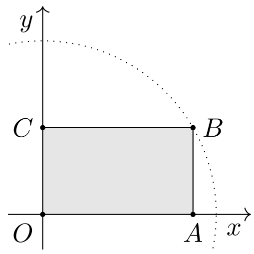
Find the rate of change of x_1 with respect to x_5.

2491. You are given that, for some positive k,

$$\int_0^k (2x-1)^2\,dx = \int_k^4 4(x-2)^3 + 6\,dx.$$

Find the value of k.

2492. Points A, B, C are at $(\cos t, 0)$, $(\cos t, \sin t)$ and $(0, \sin t)$, parametrised by $t \in [0, \pi/2]$.



Using calculus, determine the value of t for which the area of rectangle $OABC$ is stationary.

2493. Describe the transformation that takes the graph $y = (x+a)^4 + b$ onto the graph $y = (x+2a)^4 + 2b$.

2494. A child is dragging a 10 kg sledge across flat snow. The coefficient of friction is $\frac{1}{10}$, and the string is angled at 15° to the horizontal. Find the minimum force with which the child must pull, if the sledge is not to decelerate.

2495. A set of three linear equations in x and y have no simultaneous solution points (x, y). There are three distinct ways in which this can happen, which can be classified by the number n of pairwise intersections between the lines. Sketch the three cases, with

(a) $n = 3$,

(b) $n = 2$,

(c) $n = 0$.

2496. A function g is such that $\text{g}(|x|) \equiv \text{g}(x)$. Describe the symmetry of the graph $y = \text{g}(x)$.

2497. A differential equation is given as

$$\frac{dy}{dx} + 3y - e^{2x} = 0.$$

Verify that this equation is satisfied by the curve

$$y = \tfrac{1}{5}e^{-3x}(e^{5x} + 4).$$

2498. Disprove the following claim: “A quartic equation cannot have precisely three roots.”

2499. Show that, if $x^2+y^2 = 1$, then the maximum value of $x^3 + y^3$ is 1.

2500. True or false?

(a) $\dfrac{d}{dx}\,\mathrm{f}\big(x^2\big) \equiv 2x\,\mathrm{f}'\big(x^2\big)$,

(b) $\dfrac{d}{dx}\,\mathrm{f}(y) \equiv \mathrm{f}'(y)\dfrac{dy}{dx}$.

——— End of 25th Hundred ———

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